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The Role of Entanglement in Quantum Measurement and Information Processing

Paul Busch[1]

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The significance of the quantum feature of entanglement between physical systems is investigated in the context of quantum measurements. It is shown that, while there are measurement couplings that leave the object and probe systems nonentangled, no information transfer from object to probe can take place unless there is at least some intermittent period where the two systems are entangled.

KEY WORDS: quantum measurement; disturbance; entanglement; tensor product; quantum information.

1. INTRODUCTION

In recent years, the quantum feature of entanglement between physical systems has increasingly been recognized as an enormously valuable resource for purposes of information processing. Here we analyze the role of entanglement in the context of measurement processes.

In contrast to the situation in classical physics where all observables can be measured, in principle, with arbitrary accuracy and with negligible disturbance, quantum measurements are subject to the following theorem: there is no information gain without *some* state disturbance. The proof (Busch *et al.*, 1996, p. 32) is simple: assuming that the state transformer (also known as *instrument*) defined by a measurement scheme leaves *all* object states unchanged, it then follows immediately that the probabilities of measurement outcomes do not depend on the initial state of the measured object. This is to say that the measured observable is *trivial*, that is, represented by a positive operator measure whose effects are multiples of the identity operator of the underlying Hilbert space.

Hence there is no measurement with no disturbance. This disturbance is caused by the interaction that takes place between the object and the measurement device, usually mediated through a probe system. But one usually thinks of

[1] Department of Mathematics, University of Hull, Hull HU6 7RX, United Kingdom; p.busch@hull.ac.uk.

interactions producing entanglement. Hence the question arises as to whether measurements are possible which leave the object and probe in a nonentangled state, or whether there is a similar theorem to the above one that says, “no measurement without entanglement.” We will show that the answer is affirmative, but not in the most direct sense conceivable.

2. ENTANGLEMENT AND MEASUREMENT

We prove the following:

Proposition 1. *Let $\mathcal{H}_1, \mathcal{H}_2$ be complex separable Hilbert spaces, ϕ_0 a unit vector in $\mathcal{H}_2$. Assume $U : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_1 \otimes \mathcal{H}_2$ is a unitary map such that for all $\varphi \in \mathcal{H}_1$, $U(\varphi \otimes \phi_0) = \varphi' \otimes \phi'$ for some unit vectors $\varphi' \in \mathcal{H}_1$, $\phi' \in \mathcal{H}_2$. Then U acts in one of the following two ways:*

(1) *$U(\varphi \otimes \phi_0) = V(\varphi) \otimes \phi'$, where V is an isometry in $\mathcal{H}_1$ and ϕ' is a fixed unit vector in $\mathcal{H}_2$;*
(2) *$U(\varphi \otimes \phi_0) = \varphi' \otimes W_{12}\varphi$, where W_{12} is an isometry from $\mathcal{H}_1$ to $\mathcal{H}_2$ and φ' is a fixed unit vector in $\mathcal{H}_1$.*

Proof: Let $\{\varphi_n : n = 1, 2, \ldots\}$ be an orthonormal basis of $\mathcal{H}_1$. There are systems of unit vectors $\varphi'_n \in \mathcal{H}_1$, $\phi'_n \in \mathcal{H}_2$ such that $U\varphi_n \otimes \phi_0 = \varphi'_n \otimes \phi'_n$. Because of to the unitarity of U, all the vectors $\varphi'_n \otimes \phi'_n$ are mutually orthogonal. We show that one of two cases (a), (b) must hold

(a) $\{\varphi'_n\}_{n\in\mathbb{N}}$ is an orthonormal system, all ϕ'_n are parallel to ϕ'_1;
(b) $\{\phi'_n\}_{n\in\mathbb{N}}$ is an orthonormal system, all φ'_n are parallel to φ'_1.

For two vectors ψ, ξ which are mutually orthogonal, $\langle\psi|\xi\rangle = 0$, we will write $\psi \perp \xi$. If ψ, ξ are parallel, we write $\psi \parallel \xi$. Since U is unitary, this map sends orthogonal vector pairs to orthogonal pairs. Hence from $\varphi_1 \perp \varphi_2$ it follows that $\varphi'_1 \perp \varphi'_2$ or $\phi'_1 \perp \phi'_2$. Consider the first case. Then

$$U\left(\frac{1}{\sqrt{2}}(\varphi_1 + \varphi_2) \otimes \phi_0\right) = \varphi'_{12} \otimes \phi'_{12} = \frac{1}{\sqrt{2}}\varphi'_1 \otimes \phi'_1 + \frac{1}{\sqrt{2}}\varphi'_2 \otimes \phi'_2, \quad (1)$$

where $\varphi'_{12} \in \mathcal{H}_1$, $\phi'_{12} \in \mathcal{H}_2$ are some unit vectors. Since $\varphi'_1 \perp \varphi'_2$, it follows that $\phi'_2 = c\phi'_1$ with some $c \in \mathbb{C}$, $|c| = 1$. Hence we have

$$U((\varphi_1 + \varphi_2) \otimes \phi_0) = (\varphi'_1 + c\varphi'_2) \otimes \phi'_1 \quad (2)$$

Still considering the case $\varphi'_1 \perp \varphi'_2$, the relation $\varphi_2 \perp \varphi_3$ implies that $\varphi'_2 \perp \varphi'_3$ or $\phi'_2 \perp \phi'_3$. Suppose the latter holds. We show that this leads to a contradiction. Indeed this assumption gives $\varphi'_3 = c'\varphi'_2$ and thus

$$U((\varphi_1 + \varphi_2 + \varphi_3) \otimes \phi_0) = \sqrt{3}\varphi'_{123} \otimes \phi'_{123} \quad (3)$$

$$= \varphi_1' \otimes \phi_1' + \varphi_2' \otimes \phi_2' + \varphi_3' \otimes \phi_3' \tag{4}$$

$$= (\varphi_1' + c\varphi_2') \otimes \phi_1' + \varphi_3' \otimes \phi_3' \tag{5}$$

where φ_{123}' and ϕ_{123}' are some unit vectors. Recalling that $\phi_2' = c\phi_1'$ and, by assumption, $\phi_2' \perp \phi_3'$, then $\phi_1' \perp \phi_3'$, and we see that $\varphi_1' + c\varphi_2' = c''\varphi_3'$ for some $c'' \neq 0$. Upon taking the inner product of both sides with φ_1', we get (since $\varphi_1' \perp \varphi_2'$) that $\langle \varphi_1' | \varphi_1' \rangle = c'' \langle \varphi_1' | \varphi_3' \rangle = 0$) (since $\varphi_3' = c'\varphi_2' \perp \varphi_1'$). Hence $\varphi_1' = 0$ which is a contradiction.

Thus the assumption is false and we can only have $\varphi_2' \perp \varphi_3'$. Continuing inductively, we obtain that $\{\varphi_i' : i \in \mathbb{N}\}$ is an orthonormal system and all $\phi_n' = c_i \phi_1'$. Therefore, we obtain possibility (a) in the present case. Linearity then entails that $U(\varphi \otimes \phi_0) = V(\varphi) \otimes \phi_0'$ for all $\varphi \in \mathcal{H}_1$ and some isometric map V.

A completely analogous consideration can be applied in the second case of $\phi_1' \perp \phi_2'$, thus leading to the possibility (b) and

$$U(\varphi \otimes \phi_0) = \sum_i \langle \varphi_i | \varphi \rangle U(\varphi_i \otimes \phi_0)$$

$$= \varphi_0' \otimes \sum_i \langle \varphi_i | \varphi \rangle \phi_i' =: \varphi_0' \otimes W_{12}(\varphi) \tag{6}$$

for all $\varphi \in \mathcal{H}_1$ and some isometric map $W_{12} : \mathcal{H}_1 \to \mathcal{H}_2$. □

With this result we are ready to prove the following:

Theorem 1. *Let $U : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_1 \otimes \mathcal{H}_2$ be a unitary mapping such that for all vectors $\varphi \in \mathcal{H}_1$, $\phi \in \mathcal{H}_2$, the image of $\mathcal{H}_1 \otimes \mathcal{H}_2$ under U is of the form $U(\varphi \otimes \phi) = \varphi' \otimes \phi'$. Then U is one of the following:*

(a) *$U = V \otimes W$, where $V : \mathcal{H}_1 \to \mathcal{H}_1$ and $W : \mathcal{H}_2 \to \mathcal{H}_2$ are unitary;*
(b) *$U(\varphi \otimes \phi) = V_{21}\phi \otimes W_{12}\varphi$, where $V_{21} : \mathcal{H}_2 \to \mathcal{H}_1$ and $W_{12} : \mathcal{H}_1 \to \mathcal{H}_2$ are surjective isometries.*

The latter case can only occur if $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces of equal dimensions.

Proof: Let $\{\phi_i : i = 0, 1, 2, \ldots\}$ be an orthonormal basis of $\mathcal{H}_2$. For each i, we have, from Proposition 1, that either $U(\varphi \otimes \phi_i) = V_i(\varphi) \otimes \phi_i'(\star)$ with some unitary V_i or $U(\varphi \otimes \phi_i) = \varphi_i' \otimes W_{12}^{(i)}\varphi(\star\star)$ for some isometry $W_{12}^{(i)}$.

Case 1. $U(\varphi \otimes \phi_0) = V_0\varphi \otimes \phi_0'$. We show that $(\star)$ must hold for all i. Assume $(\star\star)$ holds for some $i \geq 1$. Considering the superposition $\phi_0 + \phi_i$ we find, by an argument analogous to one used in the previous proof, that either $V_0\varphi \perp \varphi_i'$ and $\phi_0' \parallel W_{12}^{(i)}\varphi$ or $\phi_0' \perp W_{12}^{(i)}\varphi$ and $V_0\varphi \parallel \varphi_i'$. The first case would violate the isometric nature of $W_{12}^{(i)}$ and the second the unitarity of V_{O}. Hence ($\star\star$) is excluded and ($\star$) must hold for all i in Case 1.

Still in Case 1, we therefore must have one of (A) $V_0\varphi \perp V_1\varphi$ and $\phi_0' \parallel \phi_1'$ or (B) $V_0\varphi \parallel V_1\varphi$ and $\phi_0' \perp \phi_1'$ Consider case (A). Two possibilities arise (considering the superposition $\phi_0 + \phi_2$): either $V_0\varphi \perp V_2\varphi$ and $\phi_0' \parallel \phi_2'$; or $V_0\varphi \parallel V_2\varphi$ and $\phi_0' \perp \phi_2'$. The second leads to $\phi_1' \perp \phi_2'$ (since $\phi_0' \parallel \phi_1'$), therefore $V_1\varphi \parallel V_2\varphi$ (considering the superposition $\phi_1 + \phi_2$) and thus $V_0\varphi \parallel V_1\varphi$, which contradicts the assumption of case (A). Hence in that case one must always have $V_0\varphi \perp V_i\varphi$ and $\phi_0' \parallel \phi_i'$ for all $i \geq 1$. Hence $\phi_i' = c_i\phi_0'$, with $|c_i| = 1$. But that implies, for $\phi = \sum_i \alpha_i\phi_i$, that $U\varphi \otimes \phi = \sum_i \alpha_i V_i\varphi \otimes \phi_i' = (\sum_i \alpha_i c_i V_i\varphi) \otimes \phi_0'$. This would contradict the surjectivity of U.

This leaves us with case (B). Suppose we have $\phi_0' \parallel \phi_2'$ and thus $V_0\varphi \perp V_2\varphi$; this give $\phi_2' \perp \phi_1'$ (from $\phi_0' \perp \phi_1'$) and so $V_1\varphi \parallel V_2\varphi$ hence $V_0\varphi \perp V_1\varphi$ (from $V_0\varphi \perp V_2\varphi$), in contradiction to (B). Therefore we must have $\phi_0' \perp \phi_2'$ and by extension of this argument, $\phi_0' \perp \phi_i'$. Furthermore, since ($\star$) holds in Case 1, similar arguments (considering superpositions $\phi_i + \phi_j$, $\phi_i + \phi_k$, $\phi_j + \phi_k$) show that we must always have $\phi_i' \perp \phi_j'$ and $V_i\varphi \parallel V_j\varphi$ for $i \neq j$. We thus obtain $V_i\varphi = c_i V_0\varphi$. It is not hard to see (considering $U((\alpha\varphi + \beta\psi) \otimes \phi_i)$ that the constants c_i are independent of φ. We get $U(\varphi \otimes \phi_i) = V_0\varphi \otimes c_i\phi_i'$. Unitarity of U enforces that V is unitary and the ϕ_i' from an orthonormal basis. Therefore we can define a unitary map W as the unique linear extension of $W\phi_i := c_i\phi_i'$. This finally leads to $U(\varphi \otimes \phi) = V_0\varphi \otimes W\phi$.

Case 2: $U(\varphi \otimes \phi_0) = \varphi_0' \otimes W_0\varphi$. Suppose $U(\varphi \otimes \phi_1) = V_1\varphi \otimes \phi_1'$. This gives either $\varphi_0' \perp V_1\varphi$ and $W_0\varphi \parallel \phi_1'$ or $\varphi_0' \parallel V_1\varphi$ and $W_0\varphi \perp \phi_1'$. Both possibilities are excluded as W_0 and V_1 (being isometric maps) do not map onto a ray. We conclude that in Case 2, $U(\varphi \otimes \phi_i) = \varphi_i' \otimes W_i\varphi$ must hold for all i.

Consider $U(\varphi \otimes \phi_1) = \varphi_1' \otimes W_1\varphi$. We must have either $\varphi_1' \perp \varphi_0'$ and $W_0\varphi \parallel W_1\varphi$, or $\varphi_1' \parallel \varphi_0'$ and $W_0\varphi \perp W_1\varphi$. In the latter case, suppose $\phi_2' \perp \phi_0'$, which goes along with $W_0\varphi \parallel W_2\varphi$. This gives $\varphi_1' \perp \varphi_2'$ and so $W_1\varphi \parallel W_2\varphi$, and therefore $W_0\varphi \parallel W_1\varphi$, in contradiction to the present case. Therefore, if $\varphi_1' \parallel \varphi_0'$ then $\varphi_i' \parallel \varphi_0'$ for all $i \geq 1$. As in Case 1, this violates the surjectivity of U.

Hence we must have the former case, $\varphi_1' \perp \varphi_0'$ and $W_0\varphi \parallel W_1\varphi$. Again in analogy to Case 1, we can conclude that $\varphi_i' \perp \varphi_j'$ and $W_i\varphi \parallel W_0\varphi$ for all i, j. We may write $W_i\varphi = c_i W_0\varphi$, where the c_i are of modulus 1 and independent of φ. Thus we get $U(\varphi \otimes \sum_i \alpha_i\phi_i) = \sum_i \alpha_i c_i \varphi_i' \otimes W_0\varphi$. Putting $W_{12} := W_0$, $\alpha_i = \langle\phi_i|\phi\rangle$, and $V_{21}\phi := \sum_i c_i\langle\phi_i|\phi\rangle\varphi_i'$, we get the final result $U(\varphi \otimes \phi) = V_{21}\phi \otimes W_{12}\varphi$. Again, unitarity of U ensures that W_0 is unitary and the φ_i' form an orthonormal basis, so that V_{12} is also unitary. □

A unitary map with the property that product states are sent to product states can be used to model dynamics that do not lead to entanglement between the systems involved. Thus it can be said that all nonentangling dynamics are of the form described in the theorem above.

Example. Let $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$. Let $E : \sum \to \mathcal{L}(\mathcal{H})$ be a positive operator valued measure (POVM) in $\mathcal{H}$, defined on a σ-algebra of subsets of some set Ω, with values in the space of bounded operators on $\mathcal{H}$. Define $U(\varphi \otimes \phi) = \phi \otimes \varphi$. Then we have

$$\langle U\varphi \otimes \phi | I \otimes E(X) U\varphi \otimes \phi\rangle = \langle \varphi | E(X)\varphi\rangle. \tag{7}$$

This is the probability reproducibility condition which makes the present model, with coupling U, pointer E, and initial probe state ϕ a measurement scheme for the observable E of the first system (Busch *et al.*, 1996, 1997).

This model demonstrates positively that information can be copied from the object onto a probe in such a way that these two systems are left nonentangled. Our theorems also show that there are two distinct types of nonentangling unitary maps: product operators or swap maps. Consider a continuous unitary group U_t which models the interaction between object and probe from time $t = 0$ to time $t = \tau$. Suppose U_t is of the form $V_t \otimes W_t$ for all t, $0 \leq t < \tau$. If U_τ were to have the form $V_{21} \otimes W_{12}$, then continuity would dictate that, as $t \to \tau$, then $V_t\varphi \to \phi$ for all φ, and $W_t\phi \to \varphi$ for all φ. But this is clearly impossible.

It follows that if a unitary continuous measurement dynamics U_t leads to a state transformation U_τ given by the swap mapping, then for $0 < t < \tau$, some of the U_t must be such that they produce entanglement; they cannot all be of product form.

3. CONCLUSION

We conclude that abstract Hilbert space quantum mechanics admits nonentangling measurements for all positive operator measures, although intermediately some entanglement must build up. Whether such measurement dynamics can be implemented by realistic interactions is another question.

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